

SPECTRAL MEASURES WITH ARBITRARY HAUSDORFF DIMENSIONS

XIN-RONG DAI AND QIYU SUN

ABSTRACT. In this paper, we consider spectral properties of Riesz product measures supported on homogeneous Cantor sets and we show the existence of spectral measures with arbitrary Hausdorff dimensions, including non-atomic zero-dimensional spectral measures and one-dimensional singular spectral measures.

1. Introduction

Given sequences $\mathcal{B} := \{b_n\}_{n=1}^\infty$ and $\mathcal{D} := \{d_n\}_{n=1}^\infty$ of positive integers that satisfy

$$(1.1) \quad 1 < d_n < b_n, \quad n = 1, 2, \dots,$$

we let

$$(1.2) \quad \rho_1 := 1 \quad \text{and} \quad \rho_n := \prod_{j=1}^{n-1} b_j \quad \text{for } n \geq 2,$$

and we define

$$(1.3) \quad C(\mathcal{B}, \mathcal{D}) := \sum_{n=1}^{\infty} \frac{\mathbb{Z}/d_n \cap [0, 1)}{\rho_n}.$$

The set $C(\mathcal{B}, \mathcal{D})$ is a *homogeneous Cantor set* contained in the interval $[0, \sum_{n=1}^{\infty} (d_n - 1)(d_n \rho_n)^{-1}]$. The reader may refer to [12, 13, 29] on homogeneous Cantor sets.

Define the Fourier transform $\hat{\mu}$ of a probability measure μ by $\hat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} d\mu(x)$. In this paper, we consider the *Riesz product measure* $\mu_{\mathcal{B}, \mathcal{D}}$

Date: December 17, 2014.

2010 Mathematics Subject Classification. 28A80, 42C05, 42C40.

Key words and phrases. spectral measure, homogenous Cantor set, Hausdorff dimension, maximal orthogonal set, maximal tree mapping, Bernoulli convolution, Beurling dimension, wavelet.

The research is partially supported by the National Science Foundation of China (No. 10871180 and 11371383), and the National Science Foundation (DMS-1109063).

defined by

$$(1.4) \quad \widehat{\mu_{\mathcal{D}, \mathcal{B}}}(\xi) := \prod_{n=1}^{\infty} H_{d_n}\left(\frac{\xi}{d_n \rho_n}\right),$$

where

$$H_m(\xi) := \frac{1}{m} \sum_{j=0}^{m-1} e^{-2\pi i j \xi} = \frac{1 - e^{-2\pi m i \xi}}{m(1 - e^{-2\pi i \xi})}, \quad m \geq 1.$$

The Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ is supported on the homogeneous Cantor set $C(\mathcal{B}, \mathcal{D})$ [12, 13], and it becomes the *Cantor measure* $\mu_{b,d}$ when $b_n = b$ and $d_n = d$ for all $n \geq 1$ [3–5, 9].

A probability measure μ with compact support is said to be a *spectral measure* if there exists a countable set Λ of real numbers, called a *spectrum*, such that $\{e^{-2\pi i \lambda x} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. A classical example of spectral measures is the Lebesgue measure on $[0, 1]$, for which the set of integers is a spectrum. Spectral properties for a probability measure are one of fundamental problems in Fourier analysis and they have close connection to tiling as formulated in Fuglede's spectral set conjecture [14, 17, 18, 20, 22, 23, 31, 32]. In 1998, Jorgensen and Pedersen [19] discovered the first families of non-atomic singular spectral measures, particularly Cantor measures $\mu_{b,2}$ with $4 \leq b \in 2\mathbb{Z}$. Since then, various singular spectral measures on self-similar/self-affine fractal sets have been found, see for instance [3–6, 9, 11, 15–17, 19, 21, 22, 24, 26, 27, 30, 34]. In this paper, we consider spectral properties of Riesz product measures $\mu_{\mathcal{B}, \mathcal{D}}$ supported on **non-self-similar** homogenous Cantor sets $C(\mathcal{B}, \mathcal{D})$.

Theorem 1.1. *Let $\mathcal{B} := \{b_n\}_{n=1}^{\infty}$ and $\mathcal{D} := \{d_n\}_{n=1}^{\infty}$ be sequences of positive integers that satisfy (1.1) and*

$$(1.5) \quad 2 \leq b_n/d_n \in \mathbb{Z} \quad \text{for all } n \geq 1.$$

Then

$$(1.6) \quad \Lambda_{\mathcal{B}, \mathcal{D}} := \bigcup_{L=1}^{\infty} \left(\sum_{n=1}^L ([0, d_n) \cap \mathbb{Z}) \rho_n \right)$$

is a spectrum of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4).

For a probability measure μ , define its Hausdorff dimension $\dim_H(\mu)$ by

$$\dim_H(\mu) := \inf \{ \dim_H(E) : \mu(E) = 1 \},$$

where $\dim_H(E)$ is the Hausdorff dimension of a set E . It is known that Cantor measures $\mu_{b,2}$ with $4 \leq b \in 2\mathbb{Z}$ have their Hausdorff dimension $\ln 2 / \ln b$. Next we estimate Hausdorff dimension of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$, with its proof given in the appendix.

Proposition 1.2. *Let $0 \leq \alpha \leq 1$, and let \mathcal{B} and \mathcal{D} be sequences of positive integers that satisfy (1.1),*

$$(1.7) \quad \lim_{n \rightarrow \infty} d_n = +\infty,$$

and

$$(1.8) \quad \lim_{n \rightarrow \infty} \frac{\ln d_n}{\ln b_n} = \alpha.$$

Then the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4) has Hausdorff dimension α ,

$$\dim_H(\mu_{\mathcal{B}, \mathcal{D}}) = \alpha.$$

Our main contribution of this paper, the existence of spectral measures with **arbitrary** Hausdorff dimension in $[0, 1]$, follows immediately from Theorem 1.1 and Proposition 1.2.

Corollary 1.3. *Let $0 \leq \alpha \leq 1$, and let \mathcal{B} and \mathcal{D} be sequences of positive integers that satisfy (1.1), (1.5), (1.7) and (1.8). Then the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4) is a spectral measure with Hausdorff dimension α .*

Taking $\alpha = 0$ and 1 in Corollary 1.3 leads to the existence of zero-dimensional non-atomic spectral measures and one-dimensional singular spectral measures respectively, cf. Fuglede's conjecture that any spectral set with positive Lebesgue measure is a tile.

2. MAXIMAL ORTHOGONAL SETS OF RIESZ PRODUCT MEASURES

A spectral measure may admit various spectra. Fourier series corresponding to different spectra could have completely different convergence rate [8, 30]. To study spectra of a probability measure μ , we recall a weak notation, *maximal orthogonal set* Λ , which means that $\{e^{-2\pi i \lambda x} : \lambda \in \Lambda\}$ is a maximal orthogonal set of $L^2(\mu)$. As Λ is a maximal orthogonal set (spectrum) of a probability measure μ if and only if its shift $\Lambda + t$ is for any real $t \in \mathbb{R}$. So in this paper we may normalize maximal orthogonal sets (spectra) by assuming that they contain the origin. In 2009, Dutkay, Han and Sun made their first attempt to characterize maximal orthogonal sets of fractal measures in [9], where a maximal orthogonal set of the one-fourth Cantor measure $\mu_{4,2}$ is labeled as a binary tree with each vertex having finite regular lengths, see [5] and references therein for general Cantor measures $\mu_{b,d}$ with $2 \leq b/d \in \mathbb{Z}$. In this section, we first consider labeling a maximal orthogonal set of the Riesz product $\mu_{\mathcal{B}, \mathcal{D}}$ on homogeneous Cantor set $C(\mathcal{B}, \mathcal{D})$.

For labeling a maximal orthogonal set, we introduce some notation. Let $\Sigma_d := \{0, 1, \dots, d-1\}$ for $d \geq 1$. For a sequence $\mathcal{D} := \{d_n\}_{n=1}^\infty$ of positive integers, let $\Sigma_{\mathcal{D}}^0 := \emptyset$, $\Sigma_{\mathcal{D}}^n := \Sigma_{d_1} \times \Sigma_{d_2} \times \dots \times \Sigma_{d_n}$ for $n \geq 1$, and $\Sigma_{\mathcal{D}}^* := \bigcup_{n=0}^\infty \Sigma_{\mathcal{D}}^n$.

be the set of all finite words. We say that a tree is a \mathcal{D} -adic tree if it has ϑ , $\Sigma_{\mathcal{D}}^n$ and $\{\delta i, i \in \Sigma_{d_{n+1}}\}$ as its root, the set of all n -th level nodes, and the set of offsprings of $\delta \in \Sigma_{\mathcal{D}}^n, n \geq 1$, respectively, where $\vartheta\delta := \delta$ and $\delta\delta'$ is the concatenation of words $\delta \in \Sigma_{d_1} \times \Sigma_{d_2} \times \cdots \times \Sigma_{d_n}$ and $\delta' \in \Sigma_{d_{n+1}} \times \Sigma_{d_{n+2}} \times \cdots \times \Sigma_{d_{n+m}}$. Given sequences $\mathcal{B} = \{b_n\}_{n=1}^\infty$ and $\mathcal{D} = \{d_n\}_{n=1}^\infty$ of positive integers satisfying (1.1), we say that $\tau : \Sigma_{\mathcal{D}}^* \rightarrow \mathbb{R}$ is a *maximal tree mapping* if

- (i) $\tau(\vartheta) = \tau(R_n(0^\infty)) = 0$ for all $n \geq 1$;
- (ii) $\tau(\delta_1 \cdots \delta_n) \in (\delta_n + d_n \mathbb{Z}) \cap \{-\lfloor b_n/2 \rfloor, -\lfloor b_n/2 \rfloor + 1, \dots, b_n - 1 - \lfloor b_n/2 \rfloor\}$ for $\delta_1 \cdots \delta_n \in \Sigma_{\mathcal{D}}^n, n \geq 1$; and
- (iii) for any word $\delta \in \Sigma_{\mathcal{D}}^n$ there exists $\delta' \in \Sigma_{d_{n+1}} \times \Sigma_{d_{n+2}} \times \cdots \times \Sigma_{d_{n+m}}$ of length $m \geq 1$ such that $\tau(R_k(\delta\delta'0^\infty)) = 0$ for sufficiently large k ,

where $0^\infty := 000 \cdots$ and $R_k(\delta) := \delta_1 \cdots \delta_k \in \Sigma_{\mathcal{D}}^k$ for $\delta = \delta_1 \cdots \delta_k \delta_{k+1} \cdots \in \bigotimes_{n=1}^\infty \Sigma_{d_n}$. For a maximal tree mapping τ , define

$$(2.1) \quad \Lambda(\tau) := \left\{ \sum_{n=1}^\infty \tau(R_n(\delta 0^\infty)) \rho_n : \delta \in \Sigma_{\mathcal{D}}^* \text{ with } \tau(R_n(\delta 0^\infty)) = 0 \text{ for sufficiently large } n \right\},$$

where $\rho_n, n \geq 1$, are given in (1.2). Following the argument used in [4, 5], we can characterize maximal orthogonal sets of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4) by maximal tree mappings.

Theorem 2.1. *Let sequences \mathcal{B} and \mathcal{D} of positive integers satisfy (1.1) and (1.5), $\mu_{\mathcal{B}, \mathcal{D}}$ be the Riesz product measure in (1.4), and for a maximal tree mapping τ let $\Lambda(\tau)$ be the set given in (2.1). Then Λ is a maximal orthogonal set of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ that contains the origin if and only if $\Lambda = \Lambda(\tau)$ for some maximal tree mapping τ .*

Denote by $\#(E)$ the cardinality of a finite set E , and define the *upper Beurling dimension* $\dim^+(\Lambda)$ of a discrete set Λ of real numbers by

$$\dim^+(\Lambda) := \inf \left\{ r > 0 : \limsup_{h \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{\#(\Lambda \cap [x - h, x + h])}{(2h)^r} < \infty \right\}.$$

Given sequences \mathcal{B} and \mathcal{D} satisfying (1.1), (1.5), (1.7) and (1.8), one may verify that the set $\Lambda(\tau)$ associated with a maximal tree mapping τ has upper Beurling dimension being less than or equal to Hausdorff dimension of the homogeneous Cantor set $C(\mathcal{B}, \mathcal{D})$,

$$(2.2) \quad \dim^+(\Lambda(\tau)) \leq \dim_H(C(\mathcal{B}, \mathcal{D})).$$

The above result is established in [10] for maximal orthogonal sets of Cantor measures $\mu_{b,d}$ with $2 \leq b/d \in \mathbb{Z}$. We remark that unlike Fourier frames on the unit interval [25], spectra of Cantor measures with zero upper Beurling dimension has been constructed by Dai, He and Lai in [5]. The reader may refer to [3–6, 8–11, 15–17, 19, 21, 22, 24–27, 29, 30, 34] and references

therein for additional information on self-similar/self-affine spectral measures.

By (2.2) and Theorem 2.1, a necessary condition for a countable set to be a spectrum of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ is that its upper Beurling dimension is less than or equal to the Hausdorff dimension of the measure $\mu_{\mathcal{B}, \mathcal{D}}$. The above necessary condition is far from being sufficient. In fact, it is a very challenging problem to find appropriate sufficient conditions, see [4, 5, 9] and references therein for recent advances. In this paper, we provide a simple sufficient condition for spectra of Riesz product measures.

Theorem 2.2. *Let $\mathcal{B} := \{b_j\}_{j=1}^\infty$ and $\mathcal{D} := \{d_j\}_{j=1}^\infty$ be sequences of positive integers that satisfy (1.1) and (1.5), τ be a maximal tree mapping and let $\Lambda = \Lambda(\tau)$ be as in (2.1). Assume that*

$$(2.3) \quad \#\{n \geq 1, \tau(R_n(\delta 0^\infty)) \neq 0\} < \infty \quad \text{for all } \delta \in \Sigma_{\mathcal{D}}^*$$

and

$$(2.4) \quad \sup_{n \geq 1} \sup_{\delta \in \Sigma_{\mathcal{D}}^n} \sum_{j=1}^{\infty} \left(\frac{|\tau(R_{n+j}(\delta 0^\infty))|}{b_{n+j}} \right)^2 < \infty,$$

then Λ is a spectrum of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4).

As an application, we have the following immediately:

Corollary 2.3. *Let $\mathcal{B}, \mathcal{D}, \tau$ and $\Lambda(\tau)$ be as in Theorem 2.2. If*

$$(2.5) \quad \sup_{n \geq 1} \sup_{\delta \in \Sigma_{\mathcal{D}}^n} \#\{j \geq 1, \tau(R_{n+j}(\delta 0^\infty)) \neq 0\} < \infty,$$

then $\Lambda(\tau)$ is a spectrum of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4).

The requirements (2.3) and (2.4) are clearly weaker than the one in (2.5), since

$$d_{n+j} \leq |\tau(R_{n+j}(\delta 0^\infty))| \leq b_{n+j}/2 \quad \text{for all } \delta \in \Sigma_{\mathcal{D}}^n \text{ and } j \geq 1.$$

We remark that those requirements are not equivalent in general when \mathcal{B} and \mathcal{D} satisfy (1.7) and (1.8) for some $0 \leq \alpha < 1$.

For sequences \mathcal{B} and \mathcal{D} satisfying (1.1) and (1.5), one may verify that the map defined by

$$\tau_{\mathcal{B}, \mathcal{D}}(\delta_1 \cdots \delta_n) := \delta_n \quad \text{for } \delta_1 \cdots \delta_n \in \Sigma_{\mathcal{D}}^n \text{ and } n \geq 0,$$

is a maximal tree mapping satisfying (2.5), and that the corresponding set $\Lambda(\tau_{\mathcal{B}, \mathcal{D}})$ is same as the spectral set $\Lambda_{\mathcal{B}, \mathcal{D}}$ in (1.6),

$$\Lambda(\tau_{\mathcal{B}, \mathcal{D}}) = \Lambda_{\mathcal{B}, \mathcal{D}}.$$

Therefore the spectral conclusion in Theorem 1.1 follows from Corollary 2.3.

3. SPECTRA OF RIESZ PRODUCT MEASURES

In this section, we prove Theorems 2.2. For that purpose, we recall a characterization about spectra of a probability measure μ with compact support, given by Jorgensen and Pederson in [19], which states that *a countable set Λ containing zero is a spectrum for $L^2(\mu)$ if and only if*

$$(3.1) \quad Q(\xi) := \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2 \equiv 1 \quad \text{for all } \xi \in \mathbb{R}.$$

Denote by $\deg(G)$ the degree of a trigonometric polynomial G . Recall that $Q(\xi)$ in (3.1) is a real analytic function. Then the proof of Theorem 2.2 reduces to establishing the following general theorem.

Theorem 3.1. *Let sequences $\mathcal{B} := \{b_n\}_{n=1}^\infty$ and $\mathcal{D} := \{d_n\}_{n=1}^\infty$ of positive integers satisfy (1.1) and (1.5), τ be a maximal tree mapping satisfying (2.3) and (2.4), and let $\Lambda(\tau)$ be as in (2.1). Assume that $\{G_n\}_{n=1}^\infty$ is a family of trigonometric polynomials satisfying $G_n(0) = 1$,*

$$(3.2) \quad \sum_{l=0}^{d_n-1} |G_n(\xi + l/d_n)|^2 = 1, \quad \xi \in \mathbb{R},$$

$$(3.3) \quad \deg(G_n) \leq D_0 d_n,$$

and

$$(3.4) \quad \inf_{d_n \xi \in [-2/3, 1/2]} |G_n(\xi)| \geq D_1,$$

where D_0, D_1 are positive constants independent of $n \geq 1$. Define a compactly supported distribution ϕ with help of its Fourier transform by

$$(3.5) \quad \hat{\phi}(\xi) := \prod_{n=1}^{\infty} G_n\left(\frac{\xi}{d_n \rho_n}\right),$$

where $\{\rho_n\}_{n=1}^\infty$ is given in (1.2). Then

$$(3.6) \quad \sum_{\lambda \in \Lambda(\tau)} |\hat{\phi}(\xi + \lambda)|^2 = 1 \quad \text{for all } \xi \in [0, 1/2].$$

Proof. Observe from (3.2) that

$$(3.7) \quad \|G_n\|_\infty := \sup_{\xi \in \mathbb{R}} |G_n(\xi)| \leq 1$$

for all $n \geq 1$. By (3.3), (3.7) and Bernstein inequality for trigonometric polynomials, we obtain that

$$(3.8) \quad |G_n(\eta/d_n) - 1| \leq \|G'_n\|_\infty |\eta/d_n| \leq D_0 \|G_n\|_\infty |\eta| \leq D_0 |\eta|$$

and

$$(3.9) \quad \begin{aligned} 0 &\leq 1 - |G_n(\eta/d_n)| \leq 1 - |G_n(\eta/d_n)|^2 \\ &\leq (\|G_n''\|_\infty \|G_n\|_\infty + \|G_n'\|_\infty^2) |\eta/d_n|^2 \leq 2D_0^2 |\eta|^2 \end{aligned}$$

for all $\eta \in [-1, 1]$ and $n \geq 1$. Thus

$$\sum_{n=1}^{\infty} \left| G_n\left(\frac{\xi}{d_n \rho_n}\right) - 1 \right| \leq D_0 \sum_{n=1}^{\infty} \rho_n^{-1} \leq D_0 \sum_{n=1}^{\infty} 4^{1-n} = \frac{4D_0}{3}$$

by (1.1), (1.5) and (3.8). Therefore the infinite product in (3.5) is well-defined and ϕ is a compactly supported distribution.

Let $C_0 > 0$ be so chosen that

$$(3.10) \quad \max(D_1, 1 - 2D_0^2 t^2) \geq \exp(-C_0 t^2) \quad \text{for all } 0 \leq t \leq 2/3.$$

Then for $L \geq 1$ and $\eta \in [-2\rho_L/3, \rho_L/2]$, we obtain from (1.1), (1.5), (3.4), (3.9) and (3.10) that

$$(3.11) \quad \left| G_L\left(\frac{\eta}{d_L \rho_L}\right) \right| \geq \max(D_1, 1 - 2D_0^2 (|\eta|/\rho_L)^2) \geq \exp(-C_0 (|\eta|/\rho_L)^2)$$

and

$$(3.12) \quad \begin{aligned} \prod_{n=L}^{\infty} \left| G_n\left(\frac{\eta}{d_n \rho_n}\right) \right| &\geq \prod_{n=L}^{\infty} \exp(-C_0 (|\eta|/\rho_n)^2) \\ &\geq \prod_{n=L}^{\infty} \exp(-C_0 (|\eta|/\rho_L)^2 \times 4^{2(n-L)}) \geq \exp(-2C_0 (|\eta|/\rho_L)^2). \end{aligned}$$

For $\xi \in [0, 1/2]$ and $\delta \in \Sigma_{\mathcal{D}}^L$, we obtained from (1.1), (1.2), (1.5) and the definition of a maximal tree mapping that

$$(3.13) \quad \xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty)) \rho_k \geq - \sum_{k=1}^L \lfloor b_k/2 \rfloor \rho_k \geq - \sum_{k=1}^L \frac{\rho_{k+1}}{2} \geq -\frac{2}{3} \rho_{L+1}$$

and

$$(3.14) \quad \xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty)) \rho_k \leq \frac{1}{2} + \sum_{k=1}^L (b_k - 1 - \lfloor b_k/2 \rfloor) \rho_k \leq \frac{1}{2} + \sum_{k=1}^L \frac{\rho_{k+1} - \rho_k}{2} \leq \frac{1}{2} \rho_{L+1}.$$

For $\delta \in \Sigma_{\mathcal{D}}^L$, let

$$\mathcal{K}(\delta) = \{k \geq 1, R_{L+k}(\delta 0^\infty) \neq 0\}.$$

For the nontrivial case that $\mathcal{K}(\delta) \neq \emptyset$, there exist finitely many positive integers $n_1 < n_2 < \dots < n_K$ by (2.3) such that

$$\mathcal{K}(\delta) = \{n_1, n_2, \dots, n_K\}.$$

Set $n_{K+1} = +\infty$ and $\lambda = \sum_{k=1}^{\infty} \tau(R_k(\delta 0^\infty))\rho_k$. Write

$$\begin{aligned} \left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right| &= |\hat{\phi}(\xi + \lambda)| \times \left(\prod_{n=L+1}^{L+n_1-1} \left| G_n\left(\frac{\xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty))\rho_k}{d_n \rho_n}\right) \right| \right)^{-1} \\ &\quad \times \left| G_{L+n_1}\left(\frac{\xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty))\rho_k}{d_{L+n_1} \rho_{L+n_1}}\right) \right|^{-1} \\ &\quad \times \left(\prod_{i=2}^K \left| G_{L+n_i}\left(\frac{\xi + \sum_{k=1}^{L+n_{i-1}} \tau(R_k(\delta 0^\infty))\rho_k}{d_{L+n_i} \rho_{L+n_i}}\right) \right| \right)^{-1} \\ &\quad \times \left(\prod_{i=1}^K \left(\prod_{n=L+n_i+1}^{L+n_{i+1}-1} \left| G_n\left(\frac{\xi + \sum_{k=1}^{L+n_i} \tau(R_k(\delta 0^\infty))\rho_k}{d_n \rho_n}\right) \right| \right) \right)^{-1} \end{aligned}$$

Then by (3.7), (3.11)–(3.14) and the definition of a maximal tree mapping, we get

$$\begin{aligned} &\left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right| \\ &\leq \exp \left(C_0 \left(\frac{2\rho_{L+1}}{3\rho_{L+n_1}} \right)^2 + C_0 \sum_{i=2}^K \left(\frac{(|\tau(R_{L+n_{i-1}}(\delta 0^\infty))| + 2/3)\rho_{L+n_{i-1}}}{\rho_{L+n_i}} \right)^2 \right) \\ &\quad \times \left(\prod_{i=1}^K \left(\prod_{n=L+n_i+1}^{\infty} \left| G_n\left(\frac{\xi + \sum_{k=1}^{L+n_i} \tau(R_k(\delta 0^\infty))\rho_k}{d_n \rho_n}\right) \right| \right) \right)^{-1} \\ &\quad \times \left(\prod_{n=L+1}^{\infty} \left| G_n\left(\frac{\xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty))\rho_k}{d_n \rho_n}\right) \right|^{-1} \right) \times |\hat{\phi}(\xi + \lambda)| \\ &\leq \exp \left(C_0 + 4C_0 \sum_{i=2}^K \left(\frac{|\tau(R_{L+n_{i-1}}(\delta 0^\infty))|}{b_{L+n_{i-1}}} \right)^2 \right) \\ &\quad \times \exp \left(2C_0 + 2C_0 \sum_{i=2}^K \left(\frac{(|\tau(R_{L+n_i}(\delta 0^\infty))| + 2/3)\rho_{L+n_i}}{\rho_{L+n_{i+1}}} \right)^2 \right) \times |\hat{\phi}(\xi + \lambda)| \\ (3.15) \leq &\exp \left(3C_0 + 12C_0 \sum_{j=1}^{\infty} \left(\frac{|\tau(R_{j+L}(\delta 0^\infty))|}{b_{j+L}} \right)^2 \right) \times |\hat{\phi}(\xi + \lambda)|, \quad \xi \in [0, 1/2]. \end{aligned}$$

For the trivial case that $\mathcal{K}(\delta) = \emptyset$,

$$\lambda := \sum_{k=1}^{\infty} \tau(R_k(\delta 0^\infty))\rho_k = \sum_{k=1}^L \tau(R_k(\delta 0^\infty))\rho_k$$

and for $\xi \in [0, 1/2]$,

$$\begin{aligned} \left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right| &= \left(\prod_{n=L+1}^{\infty} \left| G_n\left(\frac{\xi + \sum_{k=1}^L \tau(R_k(\delta 0^\infty)) \rho_k}{d_n \rho_n}\right) \right| \right)^{-1} \times |\hat{\phi}(\xi + \lambda)| \\ (3.16) \quad &\leq \exp(8C_0/9) |\hat{\phi}(\xi + \lambda)|, \end{aligned}$$

where the last inequality follows from (3.12), (3.13) and (3.14).

Define

$$\Lambda_L = \left\{ \sum_{k=1}^{\infty} \tau(R_k(\delta 0^\infty)) \rho_k : \delta \in \Sigma_{\mathcal{D}}^L \right\}, \quad L \geq 1.$$

The sets $\Lambda_L, L \geq 1$, are well-defined and satisfy

$$(3.17) \quad \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_L \rightarrow \Lambda(\tau) \text{ as } L \rightarrow +\infty$$

by (2.3). Combining (2.4), (3.15) and (3.16) leads to the existence of an absolute constant C such that

$$(3.18) \quad \left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right| \leq C |\hat{\phi}(\xi + \lambda)|$$

for all $\xi \in [0, 1/2]$ and $\lambda \in \Lambda_L$.

By (1.2), (1.5), (3.2) and the definition of a maximal tree mapping, we can prove

$$\begin{aligned} \sum_{\lambda \in \Lambda_L} \left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right|^2 &= \sum_{\delta \in \Sigma_{\mathcal{D}}^L} \left| \prod_{n=1}^L G_n\left(\frac{\xi + \sum_{k=1}^{\infty} \tau(R_k(\delta 0^\infty)) \rho_k}{d_n \rho_n}\right) \right|^2 \\ &= \sum_{\delta_1 \in \Sigma_{d_1-1}} \cdots \sum_{\delta_L \in \Sigma_{d_L-1}} \prod_{n=1}^L \left| G_n\left(\frac{\xi + \sum_{k=1}^{n-1} \tau(\delta_1 \delta_2 \cdots \delta_k) \rho_k + \delta_n \rho_n}{d_n \rho_n}\right) \right|^2 \\ &= \sum_{\delta_1 \in \Sigma_{d_1-1}} \cdots \sum_{\delta_{L-1} \in \Sigma_{d_{L-1}-1}} \prod_{n=1}^{L-1} \left| G_n\left(\frac{\xi + \sum_{k=1}^{n-1} \tau(\delta_1 \delta_2 \cdots \delta_k) \rho_k + \delta_n \rho_n}{d_n \rho_n}\right) \right|^2 \\ &\quad \times \left(\sum_{\delta_L \in \Sigma_{d_L-1}} \left| G_L\left(\frac{\xi + \sum_{k=1}^{L-1} \tau(\delta_1 \delta_2 \cdots \delta_k) \rho_k}{d_L \rho_L} + \frac{\delta_L}{d_L}\right) \right|^2 \right) \\ &= \sum_{\delta_1 \in \Sigma_{d_1-1}} \cdots \sum_{\delta_{L-1} \in \Sigma_{d_{L-1}-1}} \prod_{n=1}^{L-1} \left| G_n\left(\frac{\xi + \sum_{k=1}^{n-1} \tau(\delta_1 \delta_2 \cdots \delta_k) \rho_k + \delta_n \rho_n}{d_n \rho_n}\right) \right|^2 \\ (3.19) \quad &= \cdots = 1 \end{aligned}$$

by induction on $L \geq 1$.

By (3.5), (3.7) and (3.19), we conclude that

$$\sum_{\lambda \in \Lambda_L} |\hat{\phi}(\xi + \lambda)|^2 \leq \sum_{\lambda \in \Lambda_L} \left| \prod_{n=1}^L G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right|^2 = 1.$$

Then taking limit $L \rightarrow \infty$ in the above inequality and using (3.17) yield

$$(3.20) \quad \sum_{\lambda \in \Lambda(\tau)} |\hat{\phi}(\xi + \lambda)|^2 \leq 1.$$

Given an arbitrary $\epsilon > 0$ and $L \geq 1$, there exist an integer $M \geq L$ by (1.1), (1.5), (3.5) and (3.8) such that

$$(3.21) \quad \left| \prod_{n=1}^M G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right| \leq (1 + \epsilon) |\hat{\phi}(\xi + \lambda)|$$

for all $\xi \in [0, 1/2)$ and $\lambda \in \Lambda_L$. By (3.18), (3.19) and (3.21), we obtain that

$$\begin{aligned} 1 &= \sum_{\lambda \in \Lambda_M} \left| \prod_{n=1}^M G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right|^2 \\ &= \left(\sum_{\lambda \in \Lambda_L} + \sum_{\lambda \in \Lambda_M \setminus \Lambda_L} \right) \left| \prod_{n=1}^M G_n\left(\frac{\xi + \lambda}{d_n \rho_n}\right) \right|^2 \\ &\leq (1 + \epsilon) \sum_{\lambda \in \Lambda_L} |\hat{\phi}(\xi + \lambda)|^2 + C \sum_{\lambda \in \Lambda_M \setminus \Lambda_L} |\hat{\phi}(\xi + \lambda)|^2 \\ &\leq (1 + \epsilon) \sum_{\lambda \in \Lambda_L} |\hat{\phi}(\xi + \lambda)|^2 + C \sum_{\lambda \in \Lambda(\tau) \setminus \Lambda_L} |\hat{\phi}(\xi + \lambda)|^2. \end{aligned}$$

Taking limit $L \rightarrow +\infty$ and using (3.17) and (3.20), we have that

$$1 \leq (1 + \epsilon) \sum_{\lambda \in \Lambda(\tau)} |\hat{\phi}(\xi + \lambda)|^2 \leq 1 + \epsilon.$$

This completes the proof of the desired equation (3.6) as $\epsilon > 0$ is chosen arbitrarily. \square

We remark that trigonometric polynomials satisfying (3.2), known as *multi-channel quadrature mirror filters*, are important for the construction of multiband orthonormal wavelets [1, 33]. The requirement (3.4) could be thought as a weak version of Mallat's condition for a scaling function to have orthonormal shifts, cf. [2, 7, 28].

Trigonometric polynomials $G_n(\xi)$, $n \geq 1$, in Theorem 3.1 have factors $H_{d_n}(\xi) = \sum_{j=0}^{d_n-1} e^{-2\pi i j \xi} / d_n$ by (3.2). If $G_n(\xi)$, $n \geq 1$, are further assumed to have factors $(H_{d_n}(\xi))^N$ for some $N \geq 2$ [1, 33], then one may establish

the conclusion in Theorem 3.1 with the requirement (2.4) replaced by the following weaker assumption:

$$\sup_{n \geq 1} \sup_{\delta \in \Sigma_{\mathcal{D}}^n} \sum_{j=1}^{\infty} \left(\frac{|\tau(R_{n+j}(\delta 0^\infty))|}{b_{n+j}} \right)^{2N} < \infty.$$

APPENDIX A. Homogeneous Cantor sets

In this appendix, we evaluate Hausdorff dimensions of homogeneous Cantor sets and prove Proposition 1.2.

Given sequences $\mathcal{D} := \{d_n\}_{n=1}^\infty$ and $\mathcal{R} := \{r_n\}_{n=1}^\infty$ of positive numbers that satisfy $2 \leq d_n \in \mathbb{Z}$ and $r_n d_n \leq 1$ for all $n \geq 1$, define the *homogeneous Cantor set* $\mathcal{E}(\mathcal{R}, \mathcal{D})$ by

$$(A.1) \quad \mathcal{E}(\mathcal{R}, \mathcal{D}) := \bigcap_{n=0}^{\infty} \bigcup_{\delta \in \Sigma_{\mathcal{D}}^n} J_\delta,$$

where $\{J_\delta : \delta \in \Sigma_{\mathcal{D}}^*\}$ is the family of closed intervals contained in $J_\vartheta := [0, 1]$ such that for each $\delta \in \Sigma_{\mathcal{D}}^n, n \geq 0$, subintervals $J_{\delta k}, k \in \Sigma_{d_{n+1}}$, of J_δ satisfy the following: (i) $J_{\delta k}$ has same length $r_{n+1}|J_\delta|$ for every $k \in \Sigma_{d_{n+1}}$; (ii) the gaps between $J_{\delta k}$ and $J_{\delta(k+1)}$ have same length for all $0 \leq k < d_{n+1} - 2$; and (iii) the left endpoint of $J_{\delta 0}$ is the same as the left endpoint of J_δ , and the right endpoint of $J_{\delta(d_{n+1}-1)}$ is the same as the right endpoint of J_δ [12, 13, 29]. The above homogeneous Cantor set $\mathcal{E}(\mathcal{R}, \mathcal{D})$ has its Hausdorff dimension

$$(A.2) \quad \dim_H(\mathcal{E}(\mathcal{R}, \mathcal{D})) = \liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \ln q_j}{\sum_{j=1}^n \ln 1/r_j},$$

see for instance [13].

The set $C(\mathcal{B}, \mathcal{D})$ in (1.3) can be obtained from rescaling the homogeneous Cantor set $\mathcal{E}(\mathcal{R}, \mathcal{D})$ in (A.1). In particular,

$$(A.3) \quad C(\mathcal{B}, \mathcal{D}) = \left(\sum_{n=1}^{\infty} \frac{d_n - 1}{d_n \rho_n} \right) \mathcal{E}(\mathcal{R}, \mathcal{D})$$

with $\mathcal{R} = \{r_n\}_{n=1}^\infty$ given by

$$r_n = \frac{\sum_{j=n+1}^{\infty} (d_j - 1)/(d_j \rho_j)}{\sum_{j=n}^{\infty} (d_j - 1)/(d_j \rho_j)}, \quad n \geq 1.$$

For sequences \mathcal{B} and \mathcal{D} of positive integers satisfying (1.7) and (1.8),

$$\left| \rho_n \sum_{j=n}^{\infty} \frac{d_j - 1}{d_j \rho_j} - 1 \right| \leq \frac{1}{d_n} + \sum_{j=n+1}^{\infty} \left(\max_{i \geq n} \frac{1}{b_i} \right)^{j-n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that $\lim_{n \rightarrow \infty} r_n b_n = 1$. Combining the above limit with (1.8) and (A.2) leads to

$$(A.4) \quad \dim_H(C(\mathcal{B}, \mathcal{D})) = \lim_{n \rightarrow \infty} \frac{\sum_{j=1}^n \ln d_j}{\sum_{j=1}^n \ln 1/r_j} = \lim_{n \rightarrow \infty} \frac{\ln d_n}{\ln b_n} = \alpha.$$

Recall that the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in (1.4) is the natural measure on $C(\mathcal{B}, \mathcal{D})$. Then the Hausdorff dimension of the Riesz product measure $\mu_{\mathcal{B}, \mathcal{D}}$ in Proposition 1.2 follows from (A.4).

REFERENCES

- [1] N. BI, X.-R. DAI AND Q. SUN, *Construction of compactly supported M-band wavelets*, Appl. Comp. Harmonic Anal., 6(1999), 113-131.
- [2] A. COHEN AND Q. SUN, *An arithmetic characterization of the conjugate quadrature filters associated to orthonormal wavelet bases*, SIAM J. Math. Anal., 24(1993), 1355-1360.
- [3] X.-R. DAI, *When does a Bernoulli convolution admit a spectrum?*, Adv. Math., 231(2012), 1681-1693.
- [4] X.-R. DAI, *Spectra and maximal orthogonal sets of Cantor measures*, arXiv:1401.4630
- [5] X.-R. DAI, X.-G. HE AND C.-K. LAI, *Spectral property of Cantor measures with consecutive digits*, Adv. Math., 242(2013), 187-208.
- [6] X.-R. DAI, X.-G. HE AND K.-S. LAU, *On spectral N-Bernoulli measures*, Adv. Math., 259(2014), 511-531.
- [7] I. DAUBECHIES, *Ten Lectures on Wavelets*, SIAM, 1992.
- [8] D. DUTKAY, D. HAN AND Q. SUN, *Divergence of mock and scrambled Fourier series on fractal measures*, Trans. Amer. Math. Soc., 366(2014), 2191-2208.
- [9] D. DUTKAY, D. HAN AND Q. SUN, *On spectra of a Cantor measure*, Adv. Math., 221(2009), 251-276.
- [10] D. DUTKAY, D. HAN, Q. SUN AND E. WEBER, *On the Beurling dimension of exponential frames*, Adv. Math., 226(2011), 285-297.
- [11] D. DUTKAY AND C.-K. LAI, *Uniformity of measures with Fourier frames*, Adv. Math., 252(2014), 684-707.
- [12] K. J. FALCONER, *Fractal Geometry, Mathematical Foundations and Applications*, Wiley, New York, 1990.
- [13] D.-J. FENG, Z.-Y. WEN AND J. WU, *Some dimensional results for homogeneous Moran sets*, Sci. China, Series A, 40(1997), 475-482.
- [14] B. FUGLEDE, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal., 16(1974), 101-121.
- [15] X.-G. HE, C.-K. LAI AND K.-S. LAU, *Exponential spectra in $L^2(\mu)$* , Appl. Comput. Harmon. Anal., 34(2013), 327-338.
- [16] T.-Y. HU AND K.-S. LAU, *Spectral property of the Bernoulli convolutions*, Adv. Math., 219(2008), 554-567.
- [17] A. IOSEVICH, N. KATZ AND T. TAO, *Convex bodies with a point of curvature do not admit exponential bases*, Amer. J. Math., 123(2001), 115-120.
- [18] A. IOSEVICH, N. KATZ AND T. TAO, *Fuglede conjecture holds for convex planar domains*, Math. Res. Lett., 10(2003), 559-569.

- [19] P. JORGENSEN AND S. PEDERSEN, *Dense analytic subspaces in fractal L^2 spaces*, J. Anal. Math., 75(1998), 185-228.
- [20] M. KOLOUNTZAKIS AND M. MATOLCSI, *Tiles with no spectra*, Forum Math., 18(2006), 519-528.
- [21] I. ŁABA AND Y. WANG, *On spectral Cantor measures*, J. Funct. Anal., 193(2002), 409-420.
- [22] J. C. LAGARIAS AND Y. WANG, *Self-affine tiles in \mathbb{R}^n* , Adv. Math., 121(1996), 21-49.
- [23] J. C. LAGARIAS AND Y. WANG, *Tiling the line by the translates of one tile*, Invent. Math., 124(1996), 341-365.
- [24] C.-K. LAI, *On Fourier frame of absolutely continuous measures*, J. Funct. Anal., 261(2011), 2877-2889.
- [25] H. LANDAU, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math., 117(1967), 37-52.
- [26] J.-L. LI, $\mu_{M,D}$ -orthogonality and compatible pair, J. Funct. Anal., 244(2007), 628-638.
- [27] J.-L. LI, *Spectra of a class of self-affine measures*, J. Funct. Anal., 260(2011), 1086-1095.
- [28] S. MALLAT, *Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$* , Trans. Amer. Math. Soc., 315(1989), 69-87.
- [29] Y. PERES AND B. SOLOMYAK, *Self-similar measures and intersection of Cantor sets*, Trans. Amer. Math. Soc., 350(1998), 4065-4087.
- [30] R. S. STRICHARTZ, *Convergence of mock Fourier series*, J. Anal. Math., 99(2006), 333-353.
- [31] R. S. STRICHARTZ AND Y. WANG, *Geometry of self-affine tiles I*, Indiana Univ. Math. J., 48(1999), 1-23.
- [32] T. TAO, *Fuglede's conjecture is false in 5 or higher dimensions*, Math. Res. Lett., 11(2004), 251-258.
- [33] P. P. VAIDYNATHAN, *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.
- [34] Y. WANG, *Wavelets, tiling and spectral sets*, Duke Math. J., 114(2002), 43-57.

SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, CHINA

E-mail address: daixr@mail.sysu.edu.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816, USA

E-mail address: qiyu.sun@ucf.edu